A New Perspective on Quantum Discord Based on Quantum Measurement

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In this paper we discuss quantum correlation in terms of quantum measurement. Besides providing knowledge on the measured system, quantum measurement always introduces disturbances and decoherence on the measured system, it also results in quantum correlations between the measured system and the measuring system. For bipartite states, these measurement effects are different for the local system and the whole system. Such differences come from the quantum correlation in the bipartite state and can be used to measure it. With proper quantitative measures introduced in [Buscemi, Hayashi and Horodecki, Phys. Rev. Lett. 100, 2210504 (2008)], we show that all the differences of the above measurement effects can be unified to give a new perspective on quantum discord which is a well known quantum correlation measure.

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Quantum theory describes a quite different world from our living classical world. However, in order to make comprehensible descriptions we finally have to depart from the quantum world and come back to our classical world. Here, quantum measurement stands in and links the two conflicting theories. Quantum theory provides the microscopic world's underlying mechanism and quantum measurement provides the results for classical analysis. Fundamentally speaking, quantum theory differs from classical theory in that quantumly measurement always disturbs the measured quantum state while classically measurement can have no effect on the measured state [1]. Therefore, it is possible to make a difference between quantum theory and classical theory in terms of measurement.

Recently, many works are devoted to study the nonclassicality beyond quantum entanglement in quantum states [2–7]. Here we advocate that measurement should play a fundamental role in such a persuit. For a multipartite system, if we can find a local measurement on one subsystem which has no effect on the system's state description, then we can say the measured subsystem has only classical correlations with the others, otherwise it is inherently quantum correlated. The quantum correlation can also be quantified along this way. I.e., geometric measure may be chosen as a distance measure between the initial state and the state after measurement, which can be regarded as a nonclassicality measure [7].

In this paper we will study quantum correlations in terms of quantum measurement with the following insight. When a subsystem is quantumly correlated with the others, a local measurement on this subsystem will have different effects on the measured system and the whole system. While the measured subsystem has only classical correlations with the others as above, we can find a proper measurement which has no effects on both the measured system and the whole system, therefore their difference is zero too. In turn, the difference between the local measurement effects on the measured system and the whole system can serve as a quantum correlation measure. To move forward, we should make explicit what the effects are. It turns out that many choices can be took, measurement induced disturbance, measurement information gain, measurement induced entanglement. Although for a special measurement, the differences of the above effects may be not the same, when taking the minimum among all possible measurements which is necessary for the quantification of quantum correlation, we find that they converge to a well known quantum correlation measure: quantum discord [2, 3]. Therefore our findings also give new concrete physical meanings to quantum discord. Some interesting operational interpretations of quantum discord has been found recently [8, 9].

Quantum measurement provides information on the measured system, however at the same time it also introduces disturbance on and destroys the coherence of the measured system. The persuit of proper definitions for both of them which satisfy consistent tradeoff relation takes a long time. It was in [10] that, in terms of previous proposed concepts, Buscemi, Hayashi and Horodecki constructed both definitions and gave a sound tradeoff relation for arbitrary measurement. Here, we will make use of them for our analysis. In [15], Coles unified three viewpoints on the measurement induced decoherence. In our work, it will turn out that all these different views can be unified on quantum discord.

First we introduce the measures of information gain and disturbance of a quantum measurement constructed in [10]. A general measurement process M^B on the input system B, described by the input density matrix ρ^B on the (finite-dimensional) Hilbert space H^B, can be described as a collection of classical outcomes $X := \{x\}$,

together with a set of completely positive (CP) maps $\{E_x^B\}$, such that, when the outcome x is observed with probability $p(x) = tr[E_x^B(\rho^B)]$, the corresponding a posteriori state $\rho_x^B = E_x^B(\rho^B)/p(x)$ is output by the apparatus. Generally speaking, we can think that the action of the measurement M^B on ρ^B is given in average by the mapping $M^B(\rho^B) := \sum_x p(x)\ \rho_x^B \otimes x^X$, where $\{|x^X\rangle\}$ is a set of orthonormal (hence perfectly distinguishable) vectors on the classical register space X of outcomes. Such a measurement has an indirect measurement model in which an apparatus C is introduced to interact with B through a suitable unitary interaction and C is subsequently measured to give the classical results [11]. In this indirect model, the measured system B is separated from the final measurement made on C and the apparatus's inner degree of freedom is explicitly given. The information gain $\iota(\rho^B, M^B)$ of the measurement M^B on ρ^B is given as,

$$\iota(\rho^B, M^B) := I^{R:X}(\rho^{RX}), \tag{1}$$

the quantum disturbance introduced by measurement M^B on ρ^B is given as,

$$\begin{aligned}\delta(\rho^B, M^B) \ &: = S(\rho^B) - I_C^{R\to BX}(\rho^{RBX}) \\ &: = I^{R:CX}(\rho^{RCX}),\end{aligned} \tag{2}$$

where $I_C^{A\to B}(\sigma^{AB}) := S(\sigma^B) - S(\sigma^{AB})$ is the coherent information and R is a purifying system for which Ψ^{RB} is the purified state of density matrix ρ^B. Here, ρ^{RX} and ρ^{RBX} are the states reduced from the post measurement state ρ^{RBCX}. Eq.(1) shows that the purifying system R encodes the information while Q represents just the information carrier that is measured. Eq.(2) means that the measurement disturbance equals with the information flow into both the outputs and the internal degree of freedom of the appratus. When $\delta(\rho^B, M^B)$ is infinitely small, it is always possible to introduce a set of recovering operations $\{R_m^B\}$ which will correct the operations $\{E_x^B\}$ performed on B by the measurement. It should be noted that the correction of the measurement means that recovering operations asymptotically recover the quantum correlations between R and B. The definition of disturbance is in terms of coherent information which is closely related with decoherence. The above definitions give a balance tradeoff between information gain and quantum disturbance, $\iota(\rho^B, M^B) + \Delta(\rho^B, M^B) = \delta(\rho^B, M^B)$, here $\Delta(\rho^B, M^B) = I^{R:C|X}(\rho^{RCX})$ measures the missing information in terms of the hidden correlations between R and internal degrees of freedom of apparatus C. Now we put these concepts into a bipartite state to measure its quantum correlations.

For a bipartite state ρ^{AB} and its purified state Ψ^{RAB}, according to eq.(2) the quantum disturbance of M^B on ρ^B and on ρ^{AB} are,

$$\delta(\rho^B, M^B) = S(\rho^B) - I_C^{RA\to B'X}(\rho^{RAB'X}), \tag{3}$$

$$\delta(\rho^{AB}, M^B) = S(\rho^{AB}) - I_C^{R\to AB'X}(\rho^{RAB'X}), \tag{4}$$

their difference is $\delta(\rho^B, M^B) - \delta(\rho^{AB}, M^B) = S(\rho^B) - S(\rho^{AB}) + I_C^{R\to ABX}(\rho^{RABX}) - I_C^{RA\to BX}(\rho^{RABX}) = S(\rho^B) - S(\rho^{AB}) + S(\rho^{ABX}) - S(\rho^{BX}) = S(\rho^B) - S(\rho^{AB}) + \sum_x p_x[S(\rho_x^{AB}) - S(\rho_x^B)] \geq 0$. The information gain difference of a local measurement is $\iota(\rho^B, M^B) - \iota(\rho^{AB}, M^B) = [S(\rho^B) + H(X) - S(\rho^{BCX})] - [S(\rho^{AB}) + H(X) - S(\rho^{ABCX})] = S(\rho^B) - S(\rho^{AB}) + \sum_x p_x[S(\rho_x^{ABC}) - S(\rho_x^{BC})] \geq 0$. Through strong subaddivity of von Neumann entropy, it can be directly shown that $\delta(\rho^B, M^B) - \delta(\rho^{AB}, M^B) \geq \iota(\rho^B, M^B) - \iota(\rho^{AB}, M^B)$. All the above analysis can also be done in terms of mutual information. $\delta(\rho^B, M^B) - \delta(\rho^{AB}, M^B) = I^{RA:CX} - I^{R:CX} = I^{A:CX|R}$, $\iota(\rho^B, M^B) - \iota(\rho^{AB}, M^B) = I^{RA:X}(\rho^{RAX}) - I^{R:X}(\rho^{RX}) = I^{X:A|R}(\rho^{RAX})$, then $\delta(\rho^B, M^B) - \delta(\rho^{AB}, M^B) \geq \iota(\rho^B, M^B) - \iota(\rho^{AB}, M^B)$ follows naturally.

In order to measure the quantum correlations in ρ^{AB}, we have to find the minimum difference among all possible local measurements. We notice that, for both measurement induced disturbance and information gain, the minimum is taken for an average conditional entropy of A. Although the above two differences are different for special measurements, they converge when taking minimum among all possible measurements. The limit is a well known quantum correlation measure, quantum discord [2]. Quantum discord was initially introduced within the context of the quantum measurement problem. The definition of quantum discord [2, 3] comprises two different definitions of quantum mutual information which are extensions of two equivalent definitions of classical mutual information. Quantum mutual inforamtion $I(\rho^{AB}) = [S(\rho^A) + S(\rho^B) - S(\rho^{AB})] = S(\rho^A) - S(A|B)$, quantifies the total correlations in a bipartite state ρ^{AB}. An alternative mutual information $J_{\{\Pi_i^B\}}(\rho^{AB}) = S(\rho^A) - \sum_i p_i S(\rho_i^A) = S(\rho^A) - S(A|B_C)$, is defined based on a von Neumann measurement on party B. Here, we only consider rank-1 projective measurements, more general rank-1 POVM can be regarded as rank-1 projective measurements on extended system. Since the state after measurement is a classically correlated state $\sum_i p_i \rho_i^A \otimes \Pi_i^B$, $J_{\{\Pi_i^B\}}(\rho^{AB})$ quantifies the classical correlations in this state. Taking the difference between this two mutual informations, we obtain a quantum correlation measure $I(\rho^{AB}) - J_{\{\Pi_i^B\}}(\rho^{AB}) = S(A|B_C) - S(A|B)$, this quantity is measurement dependent. The widely used measurement indenpendent quantum discord is obtained by taking the minimum of the above quantity through all possible measurements, $D^{\leftarrow}(\rho^{AB}) = I(\rho^{AB}) - J^{\leftarrow}(\rho^{AB}) = [S(\rho^A) + S(\rho^B) - S(\rho^{AB})] - \max[S(\rho^A) - \sum_x p(x) S(\rho_x^A)] = S(\rho^B) - S(\rho^{AB}) + \min \sum_x p(x) S(\rho_x^A) = \min S(A|B_C) - S(A|B)$, here, ρ_x^A is A's state conditioned on system B's measurement output is x, $S(A|B_C)$ is A's von Neumann entropy under B's measurement. $J^{\leftarrow}(\rho^{AB})$ is the maxi-

mal mutual information or classical correlations (distillable common randomness in [12]) with one way classical communication from B to A. The last expression of quantum discord in above already hints that quantum discord can be seen as measurement induced increase on system AB's conditional information who obtains its physical interpretation in terms of quantum state merging [13]. The convergence of $\delta(\rho^B, M^B) - \delta(\rho^{AB}, M^B)$ and $\iota(\rho^B, M^B) - \iota(\rho^{AB}, M^B)$ can be obtained with rank-1 measurements of B. Hence the state of A, B and X after measurement is $\rho^{ABX} = \sum_x p(x)\rho_x^A \otimes \Pi_x^B \otimes x^X$. The reason is the concavity of conditional entropy over the convex set of POVMs and the minimum is attained on the extreme points of the set of POVMs, which are rank-1 [14]. We notice that such measurement corresponds to "Single-Kraus" or "multiplicity free" measurement which has maximal information gain or minimal disturbance [10]. Therefore, we obtain new expressions for quantum discord based on measurement effects,

$$\begin{aligned} D^{\leftarrow}(\rho^{AB}) &= \min_{M^B=\{\Pi_x^B\}}[\delta(\rho^B, M^B) - \delta(\rho^{AB}, M^B)] \\ &= \min_{M^B=\{\Pi_x^B\}}[\iota(\rho^B, M^B) - \iota(\rho^{AB}, M^B)] \end{aligned}$$

Now we make some physical discussions on the above results. When A and B are only classically correlated, it is reasonable to expect that the induced quantum disturbance of measurement M^B on A and AB should be the same since the above defined quantum disturbance $\delta(\rho^B, M^B)$ quantifies the measurement induced disturbance on the quantum coherence between the measured system and the outside world. When B is only classically correlated with A, it contributes no quantum coherence to A, hence no quantum disturbance. However, when A and B have quantum correlations between them, the situation is different. To be explicit, let us assume R to be a reference system who purifies A and B. In terms of quantum coherence, $S(\rho^B)$ quantifies the coherent interrelations between B and A, R, similarly, $S(\rho^{AB})$ quantifies the coherent interrelations between AB and R. A measurement on B introduces disturbance on both of them and we may say quantum coherence of $\delta(\rho^B, M^B)$ has been desroyed for B, at the same time quantum coherence of $\delta(\rho^{AB}, M^B)$ has been destroyed for AB. When A and B are quantum mechanically correlated, B shares part of A's coherent relations with the outsides, this part of quantum coherence of course experiences the quantum disturbance introduced by the measurement on B. However, this part of coherence does not exist in $S(\rho^{AB})$ and its disturbance naturally will not come up in $\delta(\rho^{AB}, M^B)$. In other words, the quantum disturbance $S(A|B_C) - S(A|B)$ exists between A and B, but is not perceived by C and hence is not reckoned in $\delta(\rho^{AB}, M^B)$. It may also be said that system A provides a buffer for protecting system AB against B's measurement induced disturbance or the existence of B with its quantum correlation to A retrieves $S(A|B_C) - S(A|B)$ quantum information. Because of the quantum correlation between A and B, the information gain of a measurement on B provides less information on AB than B, their minimal difference equals quantum discord.

In fact, besides disturbance and information gain, there is one other interesting effect of measurement, entanglement between the measurement apparatus and the system. For a bipartite system, the measurement induced entanglement between the apparatus and the measured system can be different from the entanglement between the apparatus and the whole system. Similarly to our results, quantum discord has already been expressed in terms of their difference, $D^{\leftarrow}(\rho^{AB}) = \min_{M^B=\{\Pi_x^B\}}[E^{M|AB} - E^{M|B}]$ [16]. We can see that for von Neumann measurements, all the above three differences becomes consistent. This point can be made clear with the following relation between conditional information and relative entropy, $S(A|B_C) - S(A|B) = D(\rho^{AB}||\sum_x \Pi_x^B\rho^{AB}\Pi_x^B) - D(\rho^B||\sum_x \Pi_x^B\rho^B\Pi_x^B)$, where $S(A|B_C) = \sum_x p(x)S(\rho_x^A)$ and $D(\rho^{AB}||\sum_x \Pi_x^B\rho^{AB}\Pi_x^B)$ corresponds to the distillable entanglement between apparatus M and AB. Noticing that $D(\rho^{AB}||\sum_x \Pi_x^B\rho^{AB}\Pi_x^B) = H^{X|R}$, where R is the purifying system of AB and $H^{X|R}$ quantifies the decoherence of B's measurement [15], the equivalence between disturbance and information gain can be found with the following equation, $H^{X|R} - H^{X|RA} = I^{RA:X}(\rho^{RAX}) - I^{R:X}(\rho^{RX})$. Therefore, we list the following expressions for quantum discord in terms of quantities with different physical meanings.

$$\begin{aligned} D^{\leftarrow}(\rho^{AB}) &= \min_{M^B=\{\Pi_x^B\}}[\delta(\rho^B, M^B) - \delta(\rho^{AB}, M^B)] \quad (5) \\ &= \min_{M^B=\{\Pi_x^B\}}[\iota(\rho^B, M^B) - \iota(\rho^{AB}, M^B)] \quad (6) \\ &= \min_{M^B=\{\Pi_x^B\}}[E^{M|AB} - E^{M|B}] \quad (7) \end{aligned}$$

It is interesting to note that, different from the initial meaning of discord which is the difference between two definitions of mutual information in quantum case, the above three equations provide another meaning for discord that is the difference between the whole and its constituents.

In conclusion, we discuss the quantum correlation in terms of measurement induced effects. The differences between the effects on the local measured part and on the whole system are used to give measures for quantum correlation. It was shown that through a minimization process which is necessary for quantum correlation measure, some differences between different effects, disturbance, information gain and measurement induced entanglement, all lead to one well known quantum correlation measure, quantum discord. Therefore, our results provide new perspective on quantum correlation in terms of measurement. Since measurement is one central con-

stituent of quantum theory and lies on the interface between quantum and classical theory, such a perspective is important and instructive.

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